Deformations of plane curves with nodes

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1 Nodes

We consider only schemes defined over a fixed algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$.

Lemma 1.1. *Let $Y \subset \mathbb{A}^2$ be a curve of equation $f(x,y) = 0$ and let $p = (\alpha, \beta) \in Y$. The following conditions are equivalent:*

(i)

$$\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \mathcal{O}_{\mathbb{A}^2,p} = (x-\alpha, y-\beta)\mathcal{O}_{\mathbb{A}^2,p} \tag{1}$$

(ii)

$$f(x+\alpha, y+\beta) = q(x,y) + \textit{higher order terms}$$

where $q(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2$ factors as a product of distinct linear forms.

Proof. Exercise. □

A point $p \in Y$ satisfying the conditions of the Lemma is called a *node*, or an *ordinary double point*, or an A_1*-singularity.* A *nodal plane curve* is a plane curve having only nodes as singularities.

A *family of affine plane curves* parametrized by an affine scheme $S = \mathrm{Spec}(R)$ (or *over* S) a morphism of the form:

$$\pi : \mathrm{Spec}\,(R[x,y]/(f)) \longrightarrow S$$

for some non-constant $f \in R[x,y]$. The morphism π is a *family of curves with nodes* (or a *family of nodal curves*) if all fibres $\mathcal{Y}(s)$ over $\mathbf{k}$-rational points $s \in S$ are nodal curves and moreover for every node $p \in \mathcal{Y}(s)$ the morphism:

$$\mathrm{Spec}\left[\mathcal{O}_{S,s}[x,y]\Big/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right] \longrightarrow \mathrm{Spec}(\mathcal{O}_{S,s})$$

is etale at p.

Example 1.2. Let $0 \neq c \in \mathbf{k}$ and consider the family of plane curves $xy + \epsilon c = 0$ over $D := \mathrm{Spec}(\mathbf{k}[\epsilon])$, where $\mathbf{k}[\epsilon] = \mathbf{k}[t]/(t^2)$. The fibre over the unique $\mathbf{k}$-rational point $(\epsilon) \in D$ is the nodal curve $xy = 0$. On the other hand:

$$\left(xy + \epsilon c, \frac{\partial(xy+\epsilon c)}{\partial x}, \frac{\partial(xy+\epsilon c)}{\partial y}\right) = (xy + \epsilon c, y, x) = (\epsilon, x, y)$$

and $\mathbf{k}[\epsilon, x, y]/(\epsilon, x, y) = \mathbf{k}$ is not flat, hence not etale, over $\mathbf{k}[\epsilon]$ (Exercise: check this). Therefore this is not a family of nodal curves.

The *constant family* $\mathrm{Spec}(\mathbf{k}[\epsilon, x, y]/(xy) \to D$ is a family of nodal curves because

$$\mathbf{k}[\epsilon, x, y]/(xy, y, x) = \mathbf{k}[\epsilon]$$

is etale over itself.

Proposition 1.3. *Let $f(\epsilon, x, y) = xy + \epsilon g(x, y) \in \mathbf{k}[\epsilon, x, y]$. Then the following conditions are equivalent:*

(a) f defines a family of nodal curves over D.

(b) $g(0, 0) = 0$.

(c) $f(\epsilon, x, y) = (x + \epsilon\alpha)(y + \epsilon\beta)$ for some $\alpha, \beta \in \mathbf{k}[x, y]$.

Proof. $(c) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$: write $g(x, y) = \alpha y + \beta x$.

$(c) \Rightarrow (a)$: define a $\mathbf{k}[\epsilon]$-automorphism $\phi : \mathbf{k}[\epsilon, x, y] \to \mathbf{k}[\epsilon, x, y]$ by

$$\phi(x) = x - \epsilon\alpha, \quad \phi(y) = y - \epsilon\beta$$

Then $\phi(f) = xy$ and therefore ϕ induces a D-isomorphism

$$\mathrm{Spec}(\mathbf{k}[\epsilon, x, y]/(f) \cong \mathrm{Spec}(\mathbf{k}[\epsilon, x, y]/(xy)$$

and (a) follows from Example 1.2.

$(a) \Rightarrow (b)$: assume by contradiction that $g(x, y) = c + \alpha y + \beta x$ with $0 \neq c \in \mathbf{k}$. define a $\mathbf{k}[\epsilon]$-automorphism $\phi : \mathbf{k}[\epsilon, x, y] \to \mathbf{k}[\epsilon, x, y]$ by

$$\phi(x) = x - \epsilon\alpha, \quad \phi(y) = y - \epsilon\beta$$

Then $\phi(f) = xy + \epsilon c$ and ϕ induces a D-isomorphism

$$\mathrm{Spec}(\mathbf{k}[\epsilon, x, y]/(f) \cong \mathrm{Spec}(\mathbf{k}[\epsilon, x, y]/(xy + \epsilon c)$$

Therefore f does not define a family of nodal curves, by Example 1.2. $\square$

2 Families of projective plane nodal curves

Let $\Sigma_d \cong \mathbb{P}^N$, where $N = \frac{d(d+3)}{2} = \binom{d+2}{2} - 1$, be the projective space parametrizing all curves of degree d in $\mathbb{P}^2$. The subset parametrizing curves having exactly δ nodes is denoted by $\mathcal{V}_{d,\delta}$.

Theorem 2.1 (Severi)**.** *$\mathcal{V}_{d,\delta}$ is a locally closed subset of Σ_d smooth of pure dimension $N - \delta$.*

$\mathcal{V}_{d,\delta}$ is called the *Severi variety* of plane curves of degree d with δ nodes. The proof of this theorem consists in introducing a sub-functor $\mathbb{V}_{d,\delta}$ of the Hilbert functor of plane curves of degree d and proving that this sub-functor is represented by a nonsingular locally closed subscheme $\mathcal{V}_{d,\delta} \subset \Sigma_d$. The definition of $\mathbb{V}_{d,\delta}$ is based on the following:

Definition 2.2. *A* family of projective plane curves of degree d with δ nodes *is a family of projective plane curves*

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook] \arrow[d, "\pi"'] & S \times \mathbb{P}^2 \arrow[dl, "pr"] \\ S & \end{tikzcd}$$

such that all fibres are curves of degree d having exactly δ distinct nodes and π is locally a family of nodal curves as defined in §1.

It is easy to show that this notion is functorial, so that we have a well defined functor:

$$\mathbb{V}_{d,\delta} : (\text{schemes}/\mathbf{k}) \longrightarrow (\text{sets})$$

by setting:

$$\mathbb{V}_{d,\delta}(S) = \{\text{families of } \delta\text{-nodal curves of degree } d \text{ over } S\}$$

The proof that $\mathbb{V}_{d,\delta}$ is representable can be found in [1], Theorem 4.7.3 p. 257. We will admit this theorem, and we will concentrate on the local properties of $\mathcal{V}_{d,\delta}$.

3 Local properties of $\mathcal{V}_{d,\delta}$

Let $C \subset \mathbb{P}^2$ be a δ-nodal plane curve of degree d and denote by $[C] \in \mathcal{V}_{d,\delta}$ the corresponding point. Let $\Delta = \{p_1, \ldots, p_\delta\}$ be the set of nodes of C. We can identify the tangent space $T_{[C]}\mathcal{V}_{d,\delta}$ with the set of families of δ-nodal curves parametrized by $\mathrm{Spec}(\mathbf{k}[\epsilon])$. Such families belong to $T_{[C]}\Sigma_d = H^0(C, \mathcal{O}_C(C))$. Let $F(X_0, X_1, X_2) = 0$ be an equation of C. An element $\overline{G} \in T_{[C]}\Sigma_d$ consists of a family of the form:

$$F + \epsilon G = 0$$

where $G = G(X_0, X_1, X_2)$ is a homogeneous polynomial of degree d representing $\overline{G} \in H^0(C, \mathcal{O}_C(C))$. From Proposition 1.3 it follows that this family defines an element of $T_{[C]}\mathcal{V}_{d,\delta}$ if and only if $G \in H^0(\mathbb{P}^2, \mathcal{I}_\Delta(d))$, or equivalently $\overline{G} \in H^0(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$. Therefore:

$$T_{[C]}\mathcal{V}_{d,\delta} = H^0(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$$

A simple local calculation shows that the obstructions to the smoothness of $\mathcal{V}_{d,\delta}$ at $[C]$ lie in $H^1(C, \mathcal{I}_\Delta \otimes \mathcal{O}_C(d))$ (Exercise).

Since $\mathcal{I}_\Delta \otimes \mathcal{O}_C(d)$ is not an invertible sheaf at the singular points we proceed as follows.

Consider the normalization $\nu : Y \to C$, and assume for simplicity that C is irreducible, so that Y is nonsingular connected of genus $g = \binom{d-1}{2} - \delta$. Let $\nu^{-1}(p_i) = x_i + y_i$, $i = 1, \ldots, \delta$. Then

$$\mathcal{I}_\Delta(d) \otimes \mathcal{O}_C = \nu_* F$$

where $F := \nu^*\mathcal{O}_C(d)(-\sum_i (x_i + y_i))$. Note that F is an invertible sheaf on Y of degree

$$d^2 - 2\delta = 2g - 2 + 3d$$

Therefore:

$$H^1(C, \mathcal{I}_\Delta(d) \otimes \mathcal{O}_C) = H^1(C, \nu_* F) = H^1(Y, F) = 0$$

and

$$h^0(C, \mathcal{I}_\Delta(d) \otimes \mathcal{O}_C) = h^0(Y, F) = N - \delta$$

This concludes the proof of Theorem 2.1.

Note: The curves defined by sections of $I_\Delta(d)$ are the so-called *adjoints* to C of degree d.

References

[1] Sernesi E.: *Deformations of Algebraic Schemes*, Springer Grundlehren b. 334 (2006).